

SIMPLIFIED A PRIORI ESTIMATE FOR THE TIME PERIODIC BURGERS' EQUATION

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ABSTRACT. We present here a version of the existence and uniqueness result of time periodic solutions to the viscous Burgers equation with irregular forcing terms (with Sobolev regularity -1 in space). The key result here is an a priori estimate which is simpler than the previously treated case of forcing terms with regularity $-\frac{1}{2}$ in time.

INTRODUCTION

The study of the Burgers' equation has a long history starting with the seminal papers by Burgers [1], Cole [2] and Hopf [7] where the Cole-Hopf transformation was introduced. The Cole-Hopf transformation transforms the homogeneous Burgers' equation into the heat equation.

More recently there have been several articles dealing with the forced Burgers' equation:

$$u_t - \nu u_{xx} + uu_x = f$$

The vast majority treats the initial value problem in time with homogeneous Dirichlet or periodic space boundary conditions (see for instance [9]).

Only recently has the question of the time-periodic forced Burgers' equation been tackled ([8, 3, 10, 4]). In most cases [8, 3] the authors are chiefly interested in the inviscid limit (the limit when the viscosity ν tends to zero).

The closest related work to ours is that of Jauslin, Kreiss and Moser [8] in which the authors show existence and uniqueness of a space and time periodic solution of the Burgers' equation for a space and time periodic forcing term which is smooth.

1. DEFINITIONS

In this section we recall some well known facts and fix some general notations.

We will be concerned with *time-periodic* solutions, of the Burgers' equation so we will use the following notation for the one-dimensional torus \mathbb{T} :

$$\mathbb{T} = \mathbb{R}/\mathbb{Z}$$

1.1. Fractional Derivatives. Given a Hilbert space H , we will denote the space of test functions with values in H by $\mathcal{D}(\mathbb{T}, H)$. Its dual space will be denoted by $\mathcal{D}'(\mathbb{T}, H^*)$:

$$\mathcal{D}'(\mathbb{T}, H^*) = (\mathcal{D}(\mathbb{T}, H))^*$$

For any positive real number s we may define the fractional derivative of order s in the following way on the space of time-periodic distributions $\mathcal{D}'(\mathbb{T}, H^*)$:

$$D^s u = \sum_{k \in \mathbb{Z}} (2\pi i k)^s u_k e^{i2\pi k t} = \sum_{k \in \mathbb{Z}} |2\pi i k|^s e^{i \operatorname{sgn}(k)s \frac{\pi}{2}} u_k e^{i2\pi k t}$$

where we have used the principal branch of the logarithm. The *sign function* is defined as follows:

$$\operatorname{sgn}(k) := \begin{cases} \frac{k}{|k|} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

For $s = 0$ we define $D^0 = \operatorname{Id}$. D^1 coincides with the usual differentiation operator on $\mathcal{D}'(\mathbb{T}, H^*)$. The familiar composition property also holds: $D^s \circ D^t = D^{s+t}$ for any $t, s \geq 0$.

The *adjoint operator* of D^s is defined by using the conjugate of the multiplier of D^s :

$$D_*^s u = \sum_{k \in \mathbb{Z}} |2\pi i k|^s e^{-i \operatorname{sgn}(k) s \frac{\pi}{2}} u_k e^{i 2\pi k t}$$

D^s and D_*^s are adjoints in the sense that for any $u \in \mathcal{D}'(\mathbb{T}, H^*)$ and $\varphi \in \mathcal{D}(\mathbb{T}, H)$:

$$\langle D^s u, \varphi \rangle = \langle u, D_*^s \varphi \rangle$$

and similarly:

$$\langle D_*^s u, \varphi \rangle = \langle u, D^s \varphi \rangle$$

1.2. Hilbert Transform. The *Hilbert transform* \mathcal{H} is defined using the multiplier $-i \operatorname{sgn}(k)$. For $u \in \mathcal{D}'(\mathbb{T}, H^*)$ let

$$\mathcal{H} u = \sum_{k \in \mathbb{Z}} -i \operatorname{sgn}(k) u_k e^{i 2\pi k t}$$

Simple computations then give:

$$D_*^{\frac{1}{2}} = D^{\frac{1}{2}} \circ \mathcal{H} = \mathcal{H} \circ D^{\frac{1}{2}}$$

Notice that if H is a function space then \mathcal{H} maps real functions to real functions. The following properties will be useful in the sequel.

$$(1) \quad \forall u \in H^{(\frac{1}{2})}(\mathbb{T}, H) \quad \left(D^{\frac{1}{2}} u, D_*^{\frac{1}{2}} \mathcal{H} u \right)_{L^2(\mathbb{T}, H)} = - \left\| D^{\frac{1}{2}} u \right\|_{L^2(\mathbb{T}, H)}^2$$

$$\forall u \in L^2(\mathbb{T}, H) \quad \Re \left((u, \mathcal{H}(u))_{L^2(\mathbb{T}, H)} \right) = 0$$

where \Re denotes the real part of the expression.

1.3. Fractional Sobolev Spaces. We define fractional Sobolev spaces in the following manner, for any $s \in \mathbb{R}$:

$$H^{(s)}(\mathbb{T}, H) = \left\{ u \in \mathcal{D}'(\mathbb{T}, H^*); \quad \sum_{k \in \mathbb{Z}} |1 + k^2|^s \|u_k\|_H^2 < \infty \right\}$$

Of course $H^{(0)}(\mathbb{T}, H) = L^2(\mathbb{T}, H)$. When $s \geq 0$ then for an $u \in L^2(\mathbb{T}, H)$: $u \in H^{(s)}(\mathbb{T}, H) \iff D^s u \in L^2(\mathbb{T}, H)$. Moreover $H^{(s)}(\mathbb{T}, H)$ is then a Hilbert space with the following scalar product:

$$(u, v) := (u, v)_{L^2(\mathbb{T}, H)} + (D^s u, D^s v)_{L^2(\mathbb{T}, H)}$$

The following classical result holds: $(H^{(s)}(\mathbb{T}, H))^* = H^{(-s)}(\mathbb{T}, H^*)$.

1.4. Anisotropic Fractional Sobolev Spaces. We will now describe some useful Sobolev spaces defined on the interval

$$I = (0, 1)$$

We will also fix a non-negative real number s :

$$s \geq 0$$

Let $H^{(s)}(I)$ denote the usual fractional Sobolev space of real-valued s -times differentiable functions on I . $H_0^{(s)}(I)$ is the closure of $\mathcal{D}(I)$ in $H^{(s)}(I)$. In that case we have $(H_0^{(s)}(I))^* = H^{(-s)}(I)$. We will also use the following notations, for α, β nonnegative real numbers:

$$H^{(\alpha)(\beta)}(\mathbb{T} \times I) := H^{(\alpha)}(\mathbb{T}, H^{(\beta)}(I))$$

and

$$H^{(\alpha, \beta)}(\mathbb{T} \times I) := H^{(\alpha)(0)}(\mathbb{T} \times I) \cap H^{(0)(\beta)}(\mathbb{T} \times I)$$

We also introduce $H_0^{(\alpha, \beta)}(\mathbb{T} \times I)$ as the closure of $\mathcal{D}(\mathbb{T} \times I)$ in $H^{(\alpha, \beta)}(\mathbb{T} \times I)$. It is clear that $H_0^{(\alpha, \beta)}(\mathbb{T} \times I) = H^{(\alpha)(0)}(\mathbb{T} \times I) \cap L^2(\mathbb{T}, H_0^{(\beta)}(I))$.

In the following sections we will use the following notations. The main working space will be $H_0^{(\frac{1}{2}, 1)}(\mathbb{T} \times I)$, for which we use the more concise notation:

$$H_0^{(\frac{1}{2}, 1)} := H_0^{(\frac{1}{2}, 1)}(\mathbb{T} \times I)$$

More generally, we will drop the dependence of the spaces on the domain $\mathbb{T} \times I$, so that L^p stands for $L^p(\mathbb{T} \times I)$, $H^{(\alpha, \beta)}$ stands for $H^{(\alpha, \beta)}(\mathbb{T} \times I)$, and so on.

We will also use the following notations, for $u \in H_0^{(\frac{1}{2}, 1)}$:

$$(2) \quad \begin{aligned} \forall u \in H_0^{(\frac{1}{2}, 1)} \quad & \tilde{u} := \mathcal{H}u \\ & u_{\sqrt{t}*} := D_*^{\frac{1}{2}}u \end{aligned}$$

2. INTERPOLATION AND REGULARITY

If $s_k(\xi)$ is the Fourier transform $s_k(\xi) = \hat{u}(k, \xi)$ of a distribution u defined on $\mathbb{T} \times \mathbb{R}$, we have the following Hölder inequality for any $\theta \in [0, 1]$:

$$\begin{aligned} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |k|^{2\alpha(1-\theta)} |\xi|^{2\beta\theta} |s_k(\xi)|^2 d\xi \leq \\ \left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |k|^{2\alpha} |s_k(\xi)|^2 d\xi \right)^{1-\theta} \left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\xi|^{2\beta} |s_k(\xi)|^2 d\xi \right)^{\theta} \end{aligned}$$

From this Hölder inequality we deduce

$$H^{(\alpha, \beta)}(\mathbb{T} \times \mathbb{R}) \hookrightarrow H^{((1-\theta)\alpha)}(\mathbb{T}, H^{(\theta\beta)}(\mathbb{R}))$$

So using an extension operator from $H^{(\theta\beta)}(I)$ to $H^{(\theta\beta)}(\mathbb{R})$ one can prove the corresponding inclusion:

$$H^{(\alpha, \beta)}(\mathbb{T} \times I) \hookrightarrow H^{((1-\theta)\alpha)(\theta\beta)}(\mathbb{T} \times I)$$

For $\alpha = 1/2$ and $\beta = 1$ and $\theta = \frac{1}{3}$ we get:

$$H_0^{(\frac{1}{2}, 1)}(\mathbb{T} \times I) \subset H^{(\frac{1}{2}, 1)}(\mathbb{T} \times I) \subset H^{(\frac{1}{3})(\frac{1}{3})}(\mathbb{T} \times I)$$

Then the vectorial Sobolev inequalities yield:

$$H_0^{(\frac{1}{2}, 1)}(\mathbb{T} \times I) \subset H^{(\frac{1}{3})(\frac{1}{3})}(\mathbb{T} \times I) \hookrightarrow L^4(\mathbb{T}, H^{(\frac{1}{3})}(I)) \hookrightarrow L^4(\mathbb{T}, L^4(I)) = L^4(\mathbb{T} \times I)$$

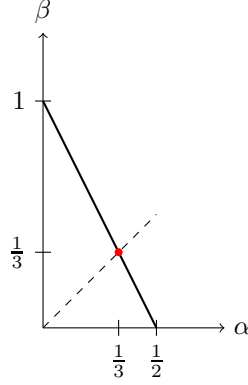


FIGURE 1. $H_0^{(\frac{1}{2},1)}$ is included in $H^{(\frac{1}{3})(\frac{1}{3})}$ which is included in L^6 by the usual Sobolev inclusion theorem. In particular, $H_0^{(\frac{1}{2},1)}$ is included in L^4 , so $u \in H_0^{(\frac{1}{2},1)} \implies u^2 \in L^2$. As a result the non-linear term of the Burgers' equation may be written as $-(u^2, v_x)$ for a test function $v \in H_0^{(\frac{1}{2},1)}$ since $v \in H_0^{(\frac{1}{2},1)} \implies v_x \in L^2$ by definition.

Here the injection $H^{(\frac{1}{3})(\frac{1}{3})}(\mathbb{T} \times I) \hookrightarrow L^4(\mathbb{T}, H^{(\frac{1}{3})})$ is compact and thus the injection $H_0^{(\frac{1}{2},1)}(\mathbb{T} \times I) \hookrightarrow L^4(\mathbb{T} \times I)$ is *compact*.

3. MAIN RESULT: A SIMPLIFIED A PRIORI ESTIMATE

We define the Burgers' Operator by:

$$T = \mathcal{L} + S$$

where \mathcal{L} and S are defined in the familiar weak form, the bracket being the *duality bracket* between $H_0^{(\frac{1}{2},1)}$ and $H^{(-\frac{1}{2},-1)}$ (recall the notations (2)):

$$\forall v \in H_0^{(\frac{1}{2},1)} \quad \langle \mathcal{L} u, v \rangle := (u_{\sqrt{t}}, v_{\sqrt{t}*}) + \mu(u_x, v_x)$$

and

$$\forall v \in H_0^{(\frac{1}{2},1)} \quad \langle S(u), v \rangle := -\frac{1}{2}(u^2, v_x)$$

It turns out that the second definition makes sense because of the embedding $H_0^{(\frac{1}{2},1)} \subset L^4$ (see [Figure 1](#)).

We now prove the main result of this article, namely an a priori estimate on the solutions of the family of equations:

$$(\mathcal{L} + \lambda S)u = f$$

for $f \in H^{(0,-1)}$ and $0 \leq \lambda \leq 1$.

In the proof we use techniques similar to those in [6]. The proof is simpler (but the result is weaker) than the one obtained in [5].

Theorem 1. *Let $f \in H^{(0,-1)}$. The set*

$$\bigcup_{\lambda \in [0,1]} (\mathcal{L} + \lambda S)^{-1}(\{f\})$$

is bounded in $H_0^{(\frac{1}{2},1)}$.

We will need the following Lemma which may be proved using a scaling argument (a proof is available in [5]).

Lemma 3.1. *There exists a constant $\mathcal{C} \in \mathbb{R}$ such that for any $u \in H_0^{(\frac{1}{2},1)}(Q)$:*

$$\int_Q |u(t, x)|^4 \, dt \, dx \leq \mathcal{C}^2 \left(\int_Q |u|^2 \, dt \, dx + \int_Q |u_{\sqrt{t}}|^2 \, dt \, dx \right) \cdot \left(\int_Q |u_x|^2 \, dt \, dx \right)$$

which implies that:

$$|u^2| \leq \mathcal{C} \|u\| |u_x|$$

Proof of Theorem 1. By definition $\mathcal{L}u + \lambda S(u) = f$ means:

$$(3) \quad \forall v \in H_0^{(\frac{1}{2},1)} \quad (u_{\sqrt{t}}, v_{\sqrt{t}*}) + \mu (u_x, v_x) - \frac{1}{2} \lambda (u^2, v_x) = \langle f, v \rangle$$

(1) We notice that for smooth u :

$$\begin{aligned} (u^2, u_x) &= \int_Q u^2 u_x \\ &= \frac{1}{3} \int_Q (u^3)_x \\ &= 0 \end{aligned}$$

and then by density and continuity this holds for all $u \in H_0^{(\frac{1}{2},1)}$.

(2) With $v = u$ in (3) we get:

$$\underbrace{(u_{\sqrt{t}}, u_{\sqrt{t}*})}_{=0} + \mu (u_x, u_x) + \frac{1}{2} \lambda \underbrace{(u^2, u_x)}_{=0} = \langle f, u \rangle$$

which gives:

$$|u_x|^2 = \frac{\langle f, u \rangle}{\mu} \leq \frac{\|f\| \|u\|}{\mu}$$

From this we deduce that

$$(4) \quad |u_x| \leq \frac{\|f\|}{\mu}$$

(3) Pairing in (3) with the Hilbert transform of u , $v = \tilde{u}$ we get:

$$(u_{\sqrt{t}}, \tilde{u}_{\sqrt{t}*}) + \mu \underbrace{(u_x, \tilde{u}_x)}_{=0} + \frac{1}{2} \lambda (u^2, \tilde{u}_x) = \langle f, \tilde{u} \rangle$$

Using the identity (1), the fact that $|\tilde{u}_x| = |u_x|$ and that $\lambda \leq 1$ we get:

$$|u_{\sqrt{t}}|^2 \leq \frac{1}{2} |(u^2, \tilde{u}_x)| + \|f\| |u_x|$$

(4) We estimate $|(u^2, \tilde{u}_x)|$ using Lemma 3.1:

$$(5) \quad \begin{aligned} |(u^2, \tilde{u}_x)| &\leq |u^2| |u_x| \\ &\leq \mathcal{C} \|u\| |u_x|^2 \end{aligned}$$

(5) Using the estimate (4) inside (5) we obtain:

$$\begin{aligned} |u_{\sqrt{t}}|^2 &\leq \frac{\mathcal{C}}{2} \|f\| |u_x|^2 + \|f\| |u_x| \\ &\leq \frac{\|f\|^2}{\mu} \left(\frac{\mathcal{C}}{2\mu} \|u\| + 1 \right) \end{aligned}$$

Since that estimate does not depend on λ the theorem is proved. \square

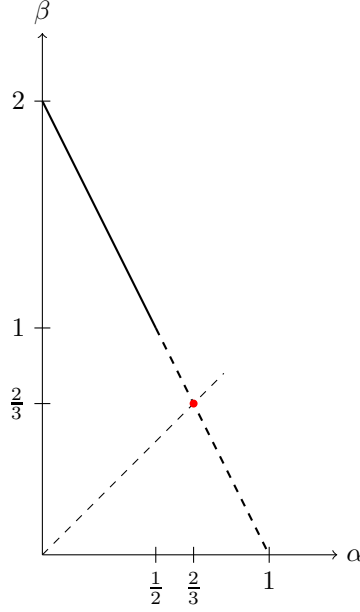


FIGURE 2. The first step of the Cole-Hopf Transformation is an integration in x . This function U obtained thus ends up in $H^{(0)(1)} \cap H^{(\frac{1}{2})(1)}$, which delimits the plain line on the graph above. But it follows from $Tu \in H^{(0)(-1)}$ that u is actually also in $H^{(1)(-1)}$ so U ends up in $H^{(1)(2)}$ and we have an inclusion in $H^{(\frac{2}{3})(\frac{2}{3})}$ which is embedded in continuous Hölder functions.

The a priori estimate above may now be used to prove existence of solutions by a (nonlinear, compact) degree argument using the Leray-Schauder Theorem (cf. [5]).

4. EXISTENCE AND UNIQUENESS OF THE TIME-PERIODIC SOLUTIONS

A weaker result of the main result proved in [5] is

Theorem 2. *For $f \in H^{(0)(-1)}$ there exists a unique solution $u \in H_0^{(\frac{1}{2},1)}$ of*

$$Tu = f$$

In this section we briefly sketch the proof of that result, the full proof being available in [5].

The main ingredient of the proof is the Cole-Hopf transformation, which is essentially defined by:

$$u = \frac{\varphi_x}{\varphi}$$

In our case there are complications due to the fact that $u \in H_0^{(\frac{1}{2},1)}$, so u is time-periodic. This change of variable will transform the periodicity problem into an eigenvalue problem (because the Cole-Hopf transformation linearises the Burgers' equation). After working out the details one shows that the uniqueness problem is equivalent to the uniqueness of the *ground state eigenvalue problem*:

Proposition 4.1. *Given $v \in H_0^{(\frac{1}{2},1)}$ the solution set of the following equation in K and φ*

$$\begin{cases} \varphi_t - \mu\varphi_{xx} + v\varphi_x + K\varphi = 0 \\ \varphi > 0 \\ \varphi_x|_{\partial Q} = 0 \\ \varphi \in H^{(1,2)} \\ K \in \mathbb{R} \end{cases}$$

is $K = 0$ and $\varphi = 1$ if and only if $Tu = Tv$ implies $u = v$ (that is, the solution to the original Burger's equation is unique).

The proof of that proposition essentially hinges on the embedding properties exposed in [section 2](#) (see [Figure 2](#)).

The remaining part of the proof is concerned with the eigenvalue problem of the Proposition above. One first shows that the eigenvalue is zero using a weaker version of the Perron-Frobenius theorem. The second step is to show that the remaining eigenvalue problem is *non degenerate*, namely that the dimension of the eigenspace must be one. This last step makes use of the a priori estimate proved in [Theorem 1](#).

The details of that part of the proof are too lengthy to be exposed here in depth so the interested reader is referred to [\[5\]](#).

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